

Q No → Obtain the Taylor and Laurent's series which represents the function $\frac{z^2-1}{(z+2)(z+3)}$ in the regions.

(i) $|z| < 2$, (ii) $2 < |z| < 3$ (iii) $|z| > 3$.

Solⁿ (i) we have

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6} = 1 - \frac{5z+7}{z^2+5z+6}$$

$$= 1 - \frac{5z+7}{(z+2)(z+3)} = 1 - \left(-\frac{3}{z+2} + \frac{8}{z+3} \right)$$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{2(1+\frac{z}{2})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{2} \left(1 + \frac{z}{2} \right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3} \right)^{-1}$$

$$= 1 + \frac{3}{2} \left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \dots \right) - \frac{8}{3} \left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \dots \right)$$

(ii) we have, $f(z) = \frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6}$

$$= 1 - \frac{5z+7}{z^2+5z+6} = 1 - \frac{5z+7}{(z+2)(z+3)} = 1 - \left(-\frac{3}{z+2} + \frac{8}{z+3} \right)$$

$$= 1 + \frac{3}{z+2} - \frac{8}{z+3} = 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z} \left(1 + \frac{2}{z} \right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3} \right)^{-1}$$

$$= 1 + \frac{3}{z} \left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \dots \right) - \frac{8}{3} \left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \dots \right)$$

(iii) We have,

$$\begin{aligned}
 f(z) &= \frac{z^2 - 1}{(z+2)(z+3)} = \frac{z^2 - 1}{z^2 + 5z + 6} \\
 &= 1 - \frac{5z + 7}{z^2 + 5z + 6} = 1 - \frac{5z + 7}{(z+2)(z+3)} \\
 &= 1 - \left(\frac{-3}{z+2} + \frac{8}{z+3} \right) = 1 + \frac{3}{z+2} - \frac{8}{z+3} \\
 &= 1 + \frac{3}{z(1 + \frac{2}{z})} - \frac{8}{z(1 + \frac{3}{z})} \\
 &= 1 + \frac{3}{z} \left(1 + \frac{2}{z} \right)^{-1} - \frac{8}{z} \left(1 + \frac{3}{z} \right)^{-1} \\
 &= 1 + \frac{3}{z} \left(1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \dots \right) - \frac{8}{z} \left(1 - \frac{3}{z} + \frac{3^2}{z^2} - \frac{3^3}{z^3} + \dots \right)
 \end{aligned}$$

~~Misc. Q. 47, 48~~

Q No → For the function $f(z) = \frac{2z^3 + 1}{z^2 + z}$, find

- (i) a Taylor Series valid in the neighbourhood of the Point $z=1$,
- (ii) a Laurent's Series valid within the annulus of which centre is the origin.

Solnⁿ

We have

$$\begin{array}{r} z^2+z \overline{) 2z^3+1} \quad \left(2z-2 \right. \\ \underline{2z^3+2z^2} \\ -2z^2+1 \\ \underline{-2z^2-2z} \\ 2z+1 \end{array}$$

$$\therefore f(z) = \frac{2z+1}{z^2+z} = \frac{2(z+1)}{z(z+1)}$$

$$= \frac{2(z-1)}{z} + \frac{1}{z+1}$$

$$= f_1(z) + f_2(z) + f_3(z) \text{ say}$$

$$\text{Where, } f_1(z) = 2(z-1)$$

$$f_2(z) = \frac{1}{z}$$

$$f_3(z) = \frac{1}{z+1}$$

$$\text{Now, } f_1(z) = f_1(i) + \sum_{n=1}^{\infty} \frac{f_1^{(n)}(i)}{n!} (z-i)^n$$

$$= 2(i-1) + 2(z-i)$$

$$\text{Again, } f_2(z) = f_2(i) + \sum_{n=1}^{\infty} \frac{f_2^{(n)}(i)}{n!} (z-i)^n$$

$$= \frac{1}{i} + \sum_{n=1}^{\infty} \frac{f_2^{(n)}(i)}{n!} (z-i)^n$$

We have,

$$f_2(z) = \frac{1}{z} \therefore f_2^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}$$

$$\therefore f_2^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$$

$$\begin{aligned}
 \therefore f_2(z) &= \frac{1}{i} + \sum_{n=1}^{\infty} \frac{(-1)^n}{i^{n+1}} (z-i)^n \\
 &= \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}} (z-i)^n
 \end{aligned}$$

Similarly, we can show that,

$$f_3(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^{n+1}}$$

Thus the Taylor's expression for $f(z)$ is given by

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}} \right] (z-i)^{n+1}$$

(ii) For $|z| < 1$, Laurent's series for $f(z)$ is given by

$$\begin{aligned}
 f(z) &= 2(z-1) + \frac{2}{z} + (1+z)^{-1} \\
 &= 2(z-1) + \frac{2}{z} + (1-z+z^2-z^3+\dots)
 \end{aligned}$$

M.V.
M. Sc
41, 92

Q No → Find the Taylor or Laurent series which represent the function.

$$\frac{1}{(z^2+1)(z+2)}$$

(i) when $|z| < 1$, (ii) when $1 < |z| < 2$ ✓

(iii) $|z| > 2$.

Solnⁿ :- (i) we have

$$\begin{aligned}f(z) &= \frac{1}{(z^2+1)(z+2)} = \frac{1}{5} \left[\frac{1}{z+2} - \frac{z-2}{z^2+1} \right] \\&= \frac{1}{5 \cdot 2 \left(1 + \frac{z}{2}\right)} - \frac{z-2}{5(1+z^2)} \\&= \frac{1}{10} \left(1 + \frac{z}{2}\right)^{-1} - \frac{z-2}{5} (1+z^2)^{-1} \\&= \frac{1}{10} \left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \dots\right) - \frac{z-2}{5} (1 - z^2 + z^4 - z^6 + \dots)\end{aligned}$$

(ii) we have,

$$\begin{aligned}f(z) &= \frac{1}{5} \left[\frac{1}{z+2} - \frac{z-2}{z^2+1} \right] \\&= \frac{1}{5 \cdot 2 \left(1 + \frac{z}{2}\right)} - \frac{z-2}{5z^2 \left(1 + \frac{1}{z^2}\right)} \\&= \frac{1}{10} \left(1 + \frac{z}{2}\right)^{-1} - \frac{z-2}{5z^2} \left(1 + \frac{1}{z^2}\right)^{-1} \\&= \frac{1}{10} \left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \dots\right) - \frac{z-2}{5z^2} \left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \dots\right)\end{aligned}$$

(iii) we have

$$\begin{aligned}f(z) &= \frac{1}{5} \left[\frac{1}{z+2} - \frac{z-2}{z^2+1} \right] \\&= \frac{1}{5 \cdot 2 \left(1 + \frac{z}{2}\right)} - \frac{z-2}{5z^2 \left(1 + \frac{1}{z^2}\right)}\end{aligned}$$

$$= \frac{1}{10z} \left(1 + \frac{z}{2}\right)^{-1} - \frac{z-2}{10z^2} \left(1 + \frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{10z} \left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \dots\right) - \frac{z-2}{10z^2} \left(1 - \frac{1}{z^2} + \dots\right)$$

$$\frac{1}{2^4} - \frac{1}{2^6} + \dots)$$

Q6, 90